

10-VERTEX GRAPHS WITH CYCLIC AUTOMORPHISM GROUP OF ORDER 4

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Abstract. We describe computational results about undirected graphs having 10 vertices and automorphism group isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

1. Introduction.

This paper deals with a special case of the problem of finding undirected graphs Γ having a given automorphism group G and minimal number of vertices. All graphs in this paper are undirected.

Studies of graph automorphism groups may be motivated by the fact that an isomorphism $G \rightarrow \text{Aut}(\Gamma)$ is a representation of abstract groups as automorphisms of symmetric binary relations in sets which can be considered as the next complexity level of discrete objects after sets. Studies of automorphism groups of graphs started in 1930s with the results of Frucht [6] who proved constructively in the late 1930s that finite graphs universally represent finite groups: for any finite group G there is a finite graph $\Gamma = (V, E)$ such that $\text{Aut}(\Gamma) \simeq G$. In the 1970s it was proved by Babai [2] constructively that for any finite group G there is a graph Γ such that $\text{Aut}(\Gamma) \simeq G$ and $|V(\Gamma)| \leq 2|G|$ if G is not cyclic of order 3, 4 or 5. An estimate $|V(\Gamma)| \leq 3|G|$ and a construction in the three exceptional cases was obtained by Sabidussi [9]. Examples of graphs with $3n$ vertices and cyclic automorphism group $\mathbb{Z}/n\mathbb{Z}$ are widely known since 1960s, see [8]. We can mention that there are 4 isomorphism types of graphs with 9 vertices which form 2 isomorphism types up to complementarity. See Babai [3] for a comprehensive exposition of this area.

It has been mentioned in the literature that 10-vertex graphs with cyclic automorphism group of order 4 do exist, see [2], [1]. There is an exercise in [7] referring to such graphs. Our goal is to summarize computational results related to this problem and popularize results of Meriwether and Arlinghaus [1].

We use standard notations of graph theory, see Diestel [5]. For a graph $\Gamma = (V, E)$ the subgraph induced by $X \subseteq V$ is denoted by $\Gamma[X]$.

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2. Main computational results.

Denote by F the set of isomorphism classes of graphs $\Gamma = (V, E)$ such that $|V| = 10$ and $\text{Aut}(\Gamma) \simeq \mathbb{Z}/4\mathbb{Z}$.

PROPOSITION 2.1. *Let $\Gamma \in F$.*

1. $|F| = 12$. Elements of F form 6 isomorphism classes up to complementarity.
2. $18 \leq |E(\Gamma)| \leq 27$.
3. $3 \leq \delta(\Gamma) \leq 5$, $4 \leq \Delta(\Gamma) \leq 6$ (minimal and maximal degree).
4. Γ has 3 $\text{Aut}(\Gamma)$ -orbits with 4, 4 and 2 vertices.
5. $\text{girth}(\Gamma) = 3$.
6. $3 \leq \omega(\Gamma) \leq 4$ (clique number).
7. $\text{core}(\Gamma)$ is isomorphic either to K_3 , K_4 or Γ .
8. $3 \leq \kappa(\Gamma) \leq 5$, $\kappa(\Gamma) = \lambda(\Gamma)$ (vertex and edge connectivity)
9. $2 \leq \text{diam}(\Gamma) \leq 3$, $\text{rad}(\Gamma) = 2$.
10. $3 \leq \chi(\Gamma) \leq 4$ (chromatic number).
11. F contains one planar graph.
12. F contains one Eulerian graph.
13. Γ is Hamiltonian.
14. Γ is not vertex, edge or distance transitive.

Proof. All statement are proved by direct computation. \square

Cases.

We describe two elements of F .

The planar graph.

The only planar graph $\Gamma_1 \in F$ is shown in Fig.1. It can be thought as embedded in the 3D space, a plane embedding is not given. $\text{Aut}(\Gamma_1)$ is generated by the vertex permutation $g = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10)$.

Subgraphs $\Gamma_1[1, 2, 3, 4, 5, 7, 9]$ and $\Gamma_1[1, 2, 3, 4, 6, 8, 10]$ which can be thought as being drawn above and below the orbit $\Gamma_1[1, 2, 3, 4]$ are interchanged by g . The core of Γ_1 is K_3 . The characteristic polynomial of Γ_1 is $(x^2 - 2)^2(x^3 - 2x^2 - 8x - 4)(x^3 +$

$$2x^2 - 4x - 4).$$

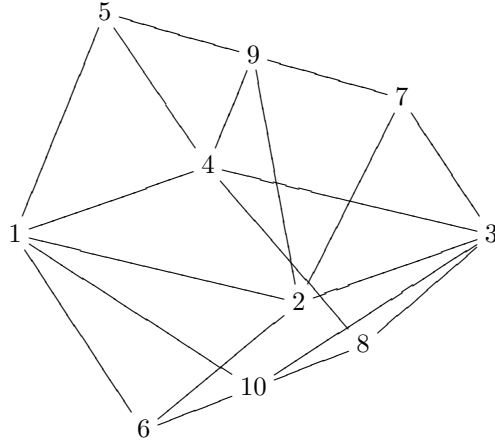


Fig.1. - Γ_1 the planar graph in F .

The graph with minimal number of edges.

The graph $\Gamma_2 \in F$ with minimal number of edges (18 edges) is shown in Fig.2. $\text{Aut}(\Gamma_2)$ is generated by the vertex permutation $g = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10)$. Γ_2 is a core.

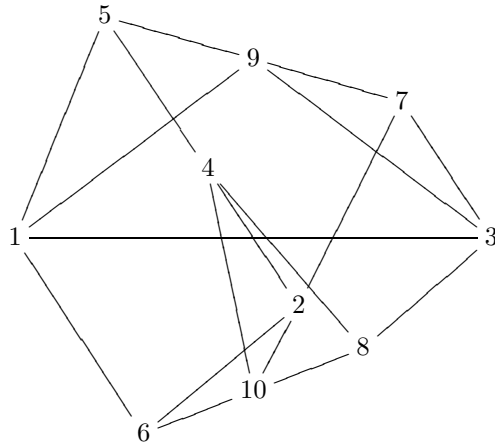


Fig.2. - Γ_2 - the graph in F with minimal number of edges.

Other graphs.

All other graphs in F can be obtained starting from Γ_1 or Γ_2 and adding or removing edges in $\Gamma_2[1, 2, 3, 4]$, the edge $(9, 10)$ and edges in $\Gamma_2[5, 6, 7, 8]$.

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